SOME PLANE TALK

Just in front of the podium here and separating me from all of you present in the clubhouse tonight there is a very large plane stretching from the floor to the ceiling and from about four feet to your right to about four feet to *my* right — that is your left of course. For the moment I will say nothing of that plane's *surface* or its *depth*. As I rotate my hand in a more or less circular fashion, from your perspective over there on your side of the plane separating us, my hand appears to be moving in a clockwise direction. Oddly enough, from my perspective on this side of the plane my hand is clearly moving in a counterclockwise direction. Were I to turn around and face a different but similar plane behind me, then we would all be on the same side of that other plane and you might notice that, even though I have continued to move my hand in the same circular fashion in which I was already moving it, you now observe that we all see my hand is moving in a counterclockwise direction. Remember this experience for a few minutes.

The planes I just mentioned are of course mathematical objects. Mathematics is one of those topics which fascinates some, frustrates others, and leaves most I suppose more or less indifferent. It is one of those fields of study which is fundamentally human. That is to say, it is part of our common heritage as human beings. For that reason alone, it is worthy of thoughtful consideration.

Many, though not all, of the oldest of human societies developed a knowledge of mathematics. In the so-called West, the oldest societies which developed a significant understanding of mathematics were the Mesopotamians and later the Egyptians and still later the Greeks. While surfing the internet once, I happened upon a photograph of a clay tablet dating from sometime between twenty-five hundred and three thousand years before the present-day so-called Common Era. The tablet depicted a figure of a circle with lines drawn inside it and some scribblings I recognized as what some pronounce cun'- e - uh- form and other pronounce cu-ne'-uh -form. I doubt any of you would be surprised to learn that I am utterly unable to decipher cuneiform. The drawn figure however made it obvious to me what the topic of the cuneiform writing was. The lines inside the circle formed a square whose corners touched the outer edge of the circle and then the square was subdivided into triangles. Clearly the nearly six thousand year old clay tablet represented a discussion of the truly ancient problem known as "squaring the circle." In effect, the anonymous ancient author of the tablet was trying to establish the relationship of the radius of the circle to its circumference. Who could imagine such a remarkably sophisticated inquiry so very long ago?

I digress albeit to note the deep fascination mathematical concepts have held for our species for a very long time indeed. As noted, the so-called "fertile crescent" was the scene of mathematical development in the West and in the so-called East, it was China and the Indian subcontinent which first nurtured serious thinking about mathematics. Self-absorbed as *our* so-called "western" civilization tends to be, we regard the zenith of civilized development of mathematics as having occurred in ancient Greece.

To be sure, the ancient Greeks did learn a great deal about mathematics over the course of just a few centuries. One group of them described itself as a school: namely the Pythagorean school. I will have more to say in a few minutes about Pythagorus, the man. For the moment however I will confine myself to his followers. As with other fields of study, the ancient Greeks characteristically took a philosophical stance in thinking about mathematics. The Pythagorean school even went so far as to theorize that everything, including human beings, actually consisted of numbers. Such a notion is clearly fanciful and at its base fundamentally mistaken. Numbers have no corporeal existence. In other words, if you should take a stroll from this spot over to the corner of Fifth and Vine, it is extremely unlikely that you would bump into any numbers on your way. Likewise, mathematical objects do not exist in so-called reality. The two planes I referred to a moment ago, one separating you from me and the other behind both you and me, were not really there, so to speak. Even though these objects did not and do not actually exist, one very real aspect of existence was demonstrated by both. Namely, they illustrated for us that being on one or the other "side" of those planes leant perspective from which we could discern the direction my hand was moving — that is, clockwise or counter-clockwise. Perhaps this feature of mathematics is what lies at the heart of the topic's fascination for us human beings. Mathematics affords us an approximation of reality from which we can deepen our understanding of the real world surrounding us.

Just in case you sense some degree of alarm at my assertion that numbers and likewise mathematical objects are not "real" in some sense, let me point out a few things which you will immediately recognize as truths. Mathematical objects are given mathematical definitions. For example, in its simplest form, a plane is defined to be a two-dimensional object. For the moment I will not test your patience by asking what it is you think you mean when you refer to a "dimension." Still. most all of us know that two dimensions might consist of say height and width. That is so very simple, isn't it?

Well then, I described the giant plane separating us a few minutes ago. Let me show you something. I am going to attempt to pierce that plane with my hand. [*Stage directions*: pass my hand towards the surface of the plane and appear to bump into it. Then, smiling broadly, slide it obviously all the way through the non-existent "entity."]

If an object by definition has only two dimensions, namely height and width, it has no depth at all. Not only can we see right through it, we can obviously also pierce it at will. In relatively advanced mathematics, one field of study is known as topology. In essence, this field of study involves how one can describe various types of surfaces with mathematical precision. When I described the imaginary plane for you earlier, I suspect all of you assumed what I had in mind was a plane that was in some sense "flat." Well it could have been but just as likely it could have been lumpy or rolled or wave-like or in some other sense NOT "flat." Think about those possibilities for a moment.

If you are prone to such imaginary speculation, think a little about how basic school-child mathematics describes a line. It is a one dimensional object. It has length and no other dimension. Now, think about such a simplistic definition of a point. It is an object in space which has no dimension at all. *What*? Can there really be such an object. It must exist because we posit that it does. But, it has absolutely no dimensions.

Well, while you dwell on that conundrum a bit, allow me to reflect on my reaction to becoming a member of this club. After joining this distinguished group a few years ago, I decided it was time to renounce my accustomed delight in the utter indolence of retirement and once again try to learn something new occasionally. When I was young, I greatly enjoyed learning about mathematics. It tickled my imagination. So, I bought a book that focused on mathematical history with the aim of seeing where my favorite subject of study came from. In keeping with the prejudices of my mid-American mid-twentieth-century birth, I assumed it all must have come from ancient Greece. Consequently, I purchased a book about the history of ancient Greek mathematics written by a Victorian Englishman. Who could be more accurate or dispassionate, right?

Not long after acquiring the book, I found myself seated in the waiting room of my local Toyota dealer as the semi-annual maintenance on my Prius was being performed. Not at all far into Sir Thomas Heath's *History of Ancient Greek Mathematics*, volume one, I began to laugh. I do not mean loud guffaws, but I really did start to smile and chuckle quietly. The good old ancient Greeks had undertaken the task of naming all sorts of different kinds of numbers. In other words, they had specified features of some numbers which set those numbers apart from other numbers. Some of the different types of numbers seem commonplace to us. Even numbers as distinguished from odd numbers, for example, seemed pretty obviously different from one another. Why that difference is significant may be a little illusory but it is nevertheless seemingly obvious. There were other distinguishing features of certain numbers that the ancient Greeks also named. So-called *prime* numbers were noted, for example. The primes have fascinated some mathematical thinkers over the centuries and indeed still do fascinate many. In fact, earlier this year one of the quadrennial Fields medals went to a young man for some entertaining work he did in the field of prime numbers. In addition to prime numbers, names were also given many other special sorts of numbers. Remember "perfect" numbers from your childhood math classes? It was Euclid, a relatively late "ancient" Greek mathematician, who identified such numbers as those whose factors, including "one," added up to the numbers themselves. For example, the number "6" has factors of one, two, and three. The sum of those factors adds up to six and thus six is indeed a perfect number. Exactly how "perfection" had anything to do with such numbers is a little hazy. However, Euclid actually did "prove" the derivation of them. Oh by the way, did you realize that every "perfect" number either ends in a six or an eight. That observation too has been mathematically "proven."

The assignment of names to various special types of numbers was the birth of what is now referred to as so-called Number Theory. Even certain *pairs* of numbers were assigned names. For example, what Heath referred to as "friendly" pairs of numbers and were later described by Steven Hawking as "amicable" pairs. "Friendly" or "amicable" pairings of numbers have fascinated some mathematicians for centuries now.

The ancient Greeks only identified one friendly or amicable pair of numbers. It is not known with certainty which ancient Greek identified that pair but some claim it was none other than Pythagorus himself. For example, Iamblichus observed that, when asked "what is a friend," Pythagorus is reported to have answered "alter ego." In English, "the other I" and then was supposed to have casually added "such as 220 and 284." This affords the basis for describing a pair of numbers as friendly or amicable when each of two numbers equals the sum of all of the aliquot parts of the other. As used here, the expression "aliquot parts" means factors, again including one. Hence, the pair of friendly or amicable numbers discovered by one or another of the ancient Greeks was 220 and 284. All right, the factors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110 and the sum of these factors is 284. Conversely, the factors of 284 are 1, 2, 4, 71, and 142 and the sum of those factors is 220.

No other mathematician identified another friendly or amicable pair of numbers until the period known as the golden age of Islamic scholarship well into the present day's so-called Current Era. Early during that golden age, Thabit ibn Qurra wrote a treatise entitled "the book of amicable numbers" in which he set out a theorem for finding such numbers. Then, in the thirteenth century of the current era, Kamel al-Din Farisi identified the pair 17,296 and 18,416. In the early seventeenth century of the current era, Muhammad Bar Yazdi disovered the pair 9,363,584 and 9,437,056. Sadly, religious animosity prevented transmission of Islamic scholarship to Europe. Hence, these two additional pairs of amicable numbers were unknown there for a long time.

Two intensely antagonistic French mathematicians, Pierre de Fermat and René Descartes, rose to the challenge. Fermat was a lawyer who only dabbled in mathematics as a hobby. Descartes, professional philosopher and mathematician, had open contempt for Fermat. Much to the annoyance of Descartes, Fermat sent a letter to a scholarly and widely respected priest known in his time as "the postbox of Europe" named Marin Mersenne in 1636 announcing his discovery of the pair 17,296 and 18,416. Descartes furiously worked to find his own new pair and in a letter to Mersenne in 1638 announced finding the pair 9,363,589 and 9,457,056.

You noticed did you not. These were indeed precisely the two pairs identified earlier by Islamic scholars. No additional pairs of friendly numbers were identified until a mathematician who lived more or less contemporaneously with good old Benjamin Franklin came up with some more. Indeed Leonhard Euler, a German-Swiss mathematician identified around sixty more pairs of friendly numbers. Strangely, in 1866 a sixteen year old boy named Nicolo Paganini discovered a previously unknown pair, 1186 and 1210. Following the advent of computers, hundreds more have been discovered.

Of what importance are amicable or friendly pars of numbers? They are of absolutely no importance whatever. The allure was not to advance human understanding of the universe by one jot. Instead, just the sport of finding them was sufficient unto itself, you might say. That very point is significant. I will recall it to your attention later.

As indicated earlier, what I have been describing to you has come to be known as "number theory." I took a course in it a couple of years ago just to see if I could still get a mental workout at a time when I was encountering a little self doubt concerning my cognitive abilities. Among the many special numbers identified by the ancient Greeks were what Heath described as "figured" numbers. In particular, they identified certain numbers as "triangular' and "square" in character, others as "rectangular" and "octagonal" and so on, even "pyramidal." Among the figured numbers, some are truly fascinating. I refer specifically to so-called "square" numbers. In relation to these particular figured numbers, the Pythagoreans identified something they referred to as "gnomens." That is spelled G-N-O-M-E-N-S in our familiar Latin alphabet. This latter term provoked a revelation for me, but more about that in a bit. I believe that what gave birth to the observations regarding figured numbers was the way in which the ancient Greeks first began to write numbers.

They derived their general style of writing numbers from styles used by earlier ancient peoples who lived in the so-called "fertile crescent." I refer to the peoples of ancient Babylon and Assyria in the region bounded more or less by the Tigris and Euphrates rivers and by extension the peoples of ancient Egypt along the banks of the Nile. All of these ancient peoples, and like them the later ancient Greeks, began writing numbers in an archaically primitive fashion. Whether impressed into a clay tablet, scratched onto a piece of papyrus, or just poked into sand with a stick, the number one was indicated by a single cypher of some distinctive shape. The number two was then depicted with an additional impression of that same cypher. Larger numbers were depicted similarly until the sheer number of cyphers may have proven confusing. In all of these cultures, either special ways of setting out cyphers or even the use of new and different cyphers allowed for the depiction of comparatively higher numbers. The important point however is that all of these cultures began by depicting relatively smaller numbers using multiple repetitions of the basic unit-cypher. We no longer do that.

Today, we use what might be described as an alphabetical-style system for writing out our numbers. That is, we have a different symbol for each of the basic set of ten numerals we write, zero through nine. I believe it was because the early Greek primitive system for writing numerals used a single cypher repeated the appropriate number of times was what led the Pythagorean school to develop the strange characterization of certain numbers as figured numbers.

Imagine sitting or standing in a group around an ancient teacher who was part of the Pythagorean school. Suppose your teacher used a stick to press a mark into some sand at your feet. "You see the number one, do you not?" he asks and we all agree. The teacher then impresses additional cyphers into the sand. Many of us notice that the added cyphers are set out or arranged in rows so that they form triangular arrays. The original cypher has two below it and then a third row appears with three cyphers and a fourth with four cyphers, all of them arranged so that they appear to form a series of triangles. At this juncture, just as Pythagorus is reported by Lucian to have instructed a student so very long ago, our teacher instructs one of us to count. The student replies "One, two, three, four." Pythagorus is reported to have interrupted his student. "Do you see? What you take for four is ten, a perfect triangle and our oath." [See Heath at page 77.] Obviously, the total number of cyphers set out in our triangular array is one plus two plus three plus four, for a sum of ten. Indeed, for the Pythagoreans the number ten was a triangular number, as were three and six. Furthermore, the number ten was regarded as the most perfect of all because of the associations described here. It recalled for the members of the school their special understanding of mathematics imparted by their founder himself.

It seems clear that triangular numbers and various other figured numbers like pentagonal numbers, pyramidal numbers, and so on may seem curiosities of a sort but are really little more than pseudo-knowledge. That is, they offer no genuine insights into special properties of numbers in general. They were observed solely as a result of the simplistic and primitive way in which the more remote ancestors of the ancients wrote down their numbers

However, there are some interesting and curious insights which can be garnered from certain categories of figured numbers noted by the Pythagoreans. For example, consider what they characterized as square numbers.

Imagine our ancient teacher poking a stick into the sand to produce a single cypher. He then draws a straight vertical line beside that one cypher and a straight horizontal line beneath it. Beside the vertical line, the teacher pokes in another cypher. Similarly, beneath the horizontal line, he pokes in yet another cypher. Finally he pokes in a cypher [use here a gesture and a sound] at the open corner to produce a figure that, when taking the first cypher into account together with the three added cyphers produces an array in the shape of a square. Now suppose the teacher again draws a vertical line this time beside the two cyphers that form an edge of the previous square containing a total of four cyphers. Similarly, he draws another horizontal line below the two cyphers along the bottom of that previous square containing a total of four cyphers. This time, the teacher pokes in two cyphers beside the added vertical line and another two beneath the added horizontal line. Finally, once again he inscribes a cypher at the empty corner [use here a gesture and a sound again], leaving a square shaped array of nine cyphers. What insights can be drawn from these square-shaped arrays?

In the field of number theory, it is common to look for series or sequences. First, notice that the number of cyphers that are inside the inscribed lines are one, then three, and by extension then five, seven, and so on. That is, they are in fact the ordinary succession of odd numbers. Now notice also that inside each new square array is the product of each successive whole number times itself. In other words, there was one times itself, or still just one. Then there were a total of two times itself or four cyphers within the square array composed of one plus the three gnomens outside the boxed-in one. Three times itself is nine and that is the number of cyphers inside the third square-shaped array containing the four cyphers inside the second square-shaped array plus the five gnomens outside the second square-shaped array. That sequence does in fact continue forever.

Now recall we noticed that the number of successive gnomens needed to yield a total of the successive numbers times themselves is the sequence of <u>*successive odd numbers*</u>. That is, we see that one plus three gnomens is the same as the product of two times itself. Similarly, we see that four cyphers in the second square array plus five gnomens is the same as the product of three times itself or a total of nine cyphers all told. Unquestionably, these observations explain why we today refer to calculating a number raised to the exponent two or in other words the product of a number times itself as *squaring* that number.

Hence we notice that two squared equals the sum of one plus three gnomens; three squared equals the sum of four plus five gnomens; four squared equals the sum of nine plus seven gnomens; and so on forever. Observations like this are at the heart of what makes number theory so satisfyingly curious.

At the heart of the foregoing observations concerning the squaring of numbers was the use of what the ancient Greeks called gnomens. That word is pointedly NOT of Greek origin. Oddly enough, it is derived from the ancient Babylonian language. For the Babylonians, a gnomon — note the difference in spelling: namely an “o - n” is used rather than an “e - n” in our

Latin script — was an astronomical instrument used to tell time by casting a shadow on a hemispherical surface. Eventually, Greek mathematicians refined uses of the term to mean, as stated by Heron of Alexandria, "that which, when added to anything, number or figure, makes the whole similar to that to which it is added." [See Heath, page 79.]

Fascinating insights regarding some of the other figured numbers, such as the rectangular or oblong numbers, were also perceived. The constraint of time precludes going into those at present.

The curious discoveries buried within the larger body of pseudo-knowledge concerning figured numbers in general which I just described illustrates how mathematics sometimes makes advances simply because someone with a quirky sort of mind notices little things about numbers that other people have overlooked. Recall the sixteen year old boy mentioned earlier who found a pair of amicable numbers. An anecdote from a relatively recent era will illustrate this fact as well.

A rather brilliant German mathematician named Carl Friederich Gauss supplies this example. Even as a child, Carl Friederich had a reputation for making calculations with remarkable speed. It is said that when little Carl Friederich was a child of around six, he had a teacher who wanted some time to undertake some personal task and so he ordered his students to add up all the whole numbers between one and one hundred. Within hardly a moment or two, little Carl Friederich lifted his hand. "Yes, Carl Friederich?," the teacher asked, expecting some question seeking clarification. However, instead of asking any clarifying question, Carl Friederich said, "Five thousand, fifty." Astonished, the teacher gasped, "Carl Friederich, you little *hannewachl*, how did you do that so quickly?" The little stinker said that he had noticed that one plus one hundred added up to one hundred one, two plus ninety-nine also added up to one hundred one, and so on until the fiftieth pair, that is the number fifty and the number fifty-one likewise added up to one hundred one. Since there were fifty such pairs of numbers which each added up to one hundred one, their sum would equal fifty times the number one hundred one, or in short five thousand fifty.

By the way, when he grew up, Carl Friederich Gauss invented an entirely new type of arithmetic. A need for it in conducting certain experiments in physics was perceived. The particular field of physics is unimportant for present purposes and I will not trouble you with describing it. I will however give you a quick and easy course in Gaussian "modal" arithmetic. It is perhaps easiest if you keep in mind the face of a clock. The number at the top of a clock face is of course twelve. In modal arithmetic, it is the top of the modem which appears there. So, for example, in "mod 4" arithmetic a four appears at the top of a clock-like face. As numbers are added or subtracted from one another, calculations are made by traveling either clockwise or counterclockwise around that face. So then, if we add "three" [here pump a finger clockwise three places around the clock-like imaginary face] plus "two" [here move the finger clockwise two further spaces clockwise around the clock-like face], we get a sum of "one." Similarly, if we take "two" [here move a finger two spaces clockwise around the clock-like face] and subtract from it the number "three" [here move the finger counterclockwise three spaces] we arrive at the number "three." Simple is it not — not to mention entertaining?

By the way, we derive our time telling pretty much directly from the ancient Babylonians. That is a little surprising is it not? The ancient Babylonians for reasons only they could understand used two separate systems for counting. One was decimal in nature like our own. The other however had a base of sixty rather than ten. When we count sixty seconds in every minute and sixty minutes in every hour, we are simply carrying on with their ancient tradition.

Why have I bored, annoyed, or maybe even entertained you with all the preceding blather tonight? Among other ancient peoples, the ancient Greeks undertook intellectual pursuits simply in order to advance their own knowledge and the sophistication of their own thought. Historians sometimes assert that the ancient Greeks were followed and indeed even conquered by the ancient Romans whose intellectual orientation was much more practical. Some say we moderns resemble more closely the Romans. To the extent that might be true, there are million dollar prizes available for the taking by any mathematician, professional or amateur, who can solve any of the so-called "millennium problems" posited by the Clay Mathematics Institute. Of course, a million dollars is no longer what it once was. For blasé mathematicians unimpressed by million-dollar prizes, however, far more substantial rewards are available. Three mathematicians from that other math and science institution up in Cambridge, Massachusetts (besides the CMI, that is) invented a method of so-called "public key encryption" which was adopted by the worldwide banking industry for transferring all sorts of financial data securely. The system was labelled RSA which combined the initials of the three MIT professors. They made a huge amount of money for their efforts. Of course, their system is no longer even in use.

Well now, why indeed *<u>have</u>* I bored, annoyed, or entertained some of you with all the preceding blather? Mathematics and indeed mathematical literature is an important part of our common heritage as human beings. If for no other reason, learning about it is important simply because it is knowledge or in rare cases pseudo knowledge. Of course, mathematics is not precisely speaking a language. Although I once attempted to argue the contrary with Bill Pratt, mathematics does not allow for anything akin to poetry. It is too precise. Of necessity, poetry requires a degree of imprecision to allow a reader to insert meaning into ambiguities intentionally created by the poet. Some have considered mathematics to be a science. Albert Einstein for example is said to have asserted that mathematics is the "perfect" science because it is constructed entirely from hypotheses that have been proven to be true. Science however asserts that there are certain predictable consequences of actions undertaken in the real world. As I believe I have shown tonight, mathematics' only relation to the real world is as a medium for *describing* the real world. Thus it can be said, as noted near the outset of this discussion, a point exists in space even though it has absolutely no dimensions solely because we *<u>posit</u>* that it exists. In a similar sense, mathematicians can — and indeed do — speak of two-dimensional globes or n-dimensional entities in space. Steven Hawking wrote a very informative book entitled *God Created The Integers*. That title was taken from a quotation Hawking liked which implied that everything contemplated by mathematics except the integers was a human creation. In truth though, even the integers were of our own devising. *<u>The entire body of mathematics has come exclusively from human minds</u>*.

<u>A CODA</u>

Tonight is just a few days after a significant anniversary. To be sure, one week ago we celebrated the anniversary of our club's founding on October 29, 1849. That date, October 29, was however also the anniversary of a much earlier event, the entrance of the Persian King Cyrus into the newly conquered city and the final, cataclysmic fall of Babylon in the year 539 BCE. Whether reading clocks and recalling our unthinking intellectual reliance on the Babylonian ancient number system or in calculating angles using a rotation of three hundred sixty degrees and doing likewise, we owe much to Babylon for some aspects of daily life whose origins we take utterly for granted. Ancient Babylon had a forgotten but profound influence on our lives millennia after its fall. So, the next time we recall how much our modern culture owes to the ancient Greeks, let us perhaps recall just a bit how much we owe to other early cultures as well.